$H-/H\infty$ fault detection observer for switched systems

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***Abstract*—This paper addresses a method for fault detection (FD) by maximizing the fault to residual sensitivity. It uses the newly developed $\mathcal{H}_-$ index properties and minimizing the well known $\mathcal{H}_\infty$ norm for worst case disturbance attenuation. The fault detection problem is formulated as LMI feasibility problem in which a cost function is minimized subject to LMI constraints. This objective is coupled to a transient response specification expressed by eigenvalue assignment formulation. This approach is then studied for both proper and strictly proper systems. Sufficient conditions are also given to enhance the disturbance decoupling. The effectiveness of the proposed approach is shown by a numerical example.**

Keywords : Residual generation, sensitivity fault detection, switched systems, $\mathcal{H}_-$ index, $\mathcal{H}_\infty$ norm, LMI.

I. Introduction

Early detection and diagnosis of process faults can help avoid abnormal event progression, reduce productivity loss and improve reliability and safety issues [1].

Among numerous fault detection techniques (FD), model based design are one popular strategy that includes observer based approach, parity-space approach [2], eigenstructure assignment approach, parameter identification based methods [3]. The idea is to compute a residual signal by comparing the mathematical model of the plant and use the relations among several measured variables to extract information on possible changes caused by faults [4].

In practical applications, the residuals are corrupted by unknown inputs such as noises, disturbances, and uncertainties in the system model. Hence, the main objective of FD methods is to generate robust residuals that are insensitive to these noise and uncertainties, while sensitive to faults. Thus, the design of a FD filter becomes a multiple objective design task [5], [6]. One interesting formulation of the problem is the $\mathcal{H}_-/\mathcal{H}_\infty$ performance.

Recent work on the $\mathcal{H}_-$ "norm" have been studied and various definition have been introduced [7]–[10]. It is the minimum "non-zero"singular value taken either at $\omega = 0$ [8], over a finite frequency range $[\underline{\omega}, \overline{\omega}]$, or over all frequency range $[0, \infty]$ [9]. The $\mathcal{H}_-$ function, which may not be a norm, is the smallest gain of a transfer matrix [11].

An interesting approach to maximize the sensibility to faults is to maximize this index. Since it is not a norm, numerous works have transformed the the $\mathcal{H}_-$ index to $\mathcal{H}_\infty$ norm by using the residual error instead of residual, then the multi-objective problem is solved using a new LMI-based filter design methodology [10].

Whilst the standard $\mathcal{H}_-/\mathcal{H}_\infty$ formulation is primarily concerned on frequency domain specification, it says a little about the transient performance. Therefore, it is very useful to combine it to the time domain performance specifications provided by eigenstructure assignment. Various eigenstructure assignment techniques have been studied and developed since the 1970's [12]–[15].

In this paper, an observer based filter is designed with the mixed $\mathcal{H}_-/\mathcal{H}_\infty$ objective with eigenvalue assignment. It can be used for both sensor and actuator faults detection. The desired observer is realized by solving a set of LMIs. A compromise between fault sensitivity and unknown input robustness is optimized via a convex optimization algorithm.

The outline of this paper is as follows. After the Introduction, problem formulation is given in Section II. In section III, preliminaries for the synthesis of $\mathcal{H}_\infty$ observer, $\mathcal{H}_-$ fault detector, eigen assignment and conditions for disturbance decoupling. Fault detection observer scheme is given in Section IV using additive filter design, for both proper and strictly proper systems. A min/max criterion is used to solve an optimization problem set by the LMIs. The above results are illustrated by a numerical example in Section V. Finally, Section VI shows the concluding remarks and the possible future work.

Notations: The notation used in this paper is standard. X^T is the transposed of matrix X, the star symbol $(\star)$ in a symmetric matrix denotes the transposed block in the symmetric position. The notation $P > (<)0$ means P is real symmetric positive (negative) definite matrix. 0 and I denote zeros and identity matrix of appropriate dimensions.

II. Problem Formulation

Consider the state space representation of the switched linear time invariant system :

$$\begin{aligned} \dot{x}(t) &= A_{\alpha(t)}x(t) + B_{\alpha(t)}u(t) \\ &\quad + E_{d,\alpha(t)}d(t) + E_{f,\alpha(t)}f(t) \qquad (1) \\ y(t) &= C_{\alpha(t)}x(t) + D_{\alpha(t)}u(t) \\ &\quad + F_{d,\alpha(t)}d(t) + F_{f,\alpha(t)}f(t) \qquad (2) \end{aligned}$$

where $x \in \mathbb{R}^n$ is the state vector, $y \in \mathbb{R}^p$ is the measurement output vector, $u \in \mathbb{R}^m$ is the input vector, $d \in \mathbb{R}^{n_d}$ is the disturbance vector, $f \in \mathbb{R}^{n_f}$ is the vector of faults to be detected, $\alpha(t)$ is the switching signal, it is assumed known

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and measured. The matrices A_α, B_α, $E_{d,\alpha}$, $E_{f,\alpha}$, C_α, D_α, $F_{d,\alpha}$ and $F_{f,\alpha}$ are known and in appropriate dimensions. In the following the subscript t is omitted without confusion for typing simplifications.

The identity observer used by the residual generator is:

$$\dot{\hat{x}} = A_\alpha \hat{x} + B_\alpha u + L_\alpha(y - \hat{y}) \quad (3)$$
$$\hat{y} = C_\alpha \hat{x} + D_\alpha u \quad (4)$$
$$\tilde{r}_\alpha = y - \hat{y}, \quad r_\alpha = Q_\alpha \tilde{r}_\alpha \quad (5)$$

The state error $\tilde{x} = x - \hat{x}$ then:

$$\begin{aligned} \dot{\tilde{x}} &= (A_\alpha - L_\alpha C_\alpha)\tilde{x} + (E_{d,\alpha} - L_\alpha F_{d,\alpha})d \\ &\quad + (E_{f,\alpha} - L_\alpha F_{f,\alpha})f \quad (6) \\ r_\alpha &= Q_\alpha C_\alpha \tilde{x} + Q_\alpha F_{d,\alpha} d + Q_\alpha F_{f,\alpha} f \quad (7) \end{aligned}$$

The state space representation of the observer has the form:

$$\dot{\tilde{x}} = A^*_\alpha \tilde{x} + B^*_{d,\alpha} d + B^*_{f,\alpha} f \quad (8)$$
$$\tilde{r}_\alpha = C_\alpha \tilde{x} + F_{d,\alpha} d + F_{f,\alpha} f \quad (9)$$
$$r_\alpha = Q_\alpha C_\alpha \tilde{x} + Q_\alpha F_{d,\alpha} d + Q_\alpha F_{f,\alpha} f \quad (10)$$

with $A^*_\alpha = A_\alpha - L_\alpha C_\alpha$, $B^*_{d,\alpha} = E_{d,\alpha} - L_\alpha F_{d,\alpha}$ and $B^*_{f,\alpha} = E_{f,\alpha} - L_\alpha F_{f,\alpha}$.

Let the sensitivity functions of fault and disturbance to the residual be:

$$T_{rf_\alpha}(s) = Q_\alpha C_\alpha (sI - A^*_\alpha)^{-1} B^*_{f,\alpha} + Q_\alpha F_{f,\alpha} \quad (11)$$
$$T_{rd_\alpha}(s) = Q_\alpha C_\alpha (sI - A^*_\alpha)^{-1} B^*_{d,\alpha} + Q_\alpha F_{d,\alpha} \quad (12)$$

The objective of the $\mathcal{H}_-/\mathcal{H}_\infty$ switched FD observer is resumed by the following conditions:

$$\|T_{rd_\alpha}\|_2 < \gamma_\alpha \|d\|_2 \quad (13)$$
$$\|T_{rf_\alpha}\|_2 > \beta_\alpha \|f\|_2 \quad (14)$$

The problem is formulated as following: Find matrices L_α and Q_α that maximize β_α and minimize γ_α such that the switched FD observer is stable. The optimization criterion used in this paper is to maximize $\beta^2_\alpha - \gamma^2_\alpha$.

Assumption 1: In this study the pair (A_α, C_α) is assumed observable, or without loss of generality is detectable. It is a standard assumption for all fault detection problems.

III. Preliminaries

Definition 1: The $\mathcal{H}_\infty$ norm of a transfer function $G(s)$ is defined as [16]:

$$\|G(s)\|_\infty = \sup_{\omega \in \mathbb{R}} \overline{\sigma}(G(j\omega)) \quad (15)$$

where $\overline{\sigma}$ denotes the maximum singular value of $G(s)$.

Definition 2: The $\mathcal{H}_-$ index of a transfer function $G(s)$ is defined as [17]:

$$\|G(s)\|_-^{[0,\overline{\omega}]} = \inf_{\omega \in [0,\overline{\omega}]} \underline{\sigma}(G(j\omega)) \quad (16)$$

where $\underline{\sigma}$ denotes the minimum singular value .
The $\mathcal{H}_-$ index is neither a norm nor a matrix norm. However it gives a suitable definition for minimum sensitivity in fault detection problems.

Theorem 1: Consider the fault detection observer in (9)-(10). The following expressions are equivalent:

1) There exist $\gamma_\alpha \in \mathbb{R}, \gamma_\alpha > 0$ such that the inequality holds:
$$\|r_{d_\alpha}\|_2 < \gamma_\alpha \|d\|_2 \quad (17)$$
2) There exist $\gamma_\alpha \in \mathbb{R}$, $\gamma_\alpha > 0$ such that the inequality holds:
$$\|T_{rd_\alpha}(s)\|_\infty < \gamma_\alpha \quad (18)$$
3) (Bounded real lemma) There exists a matrix L_α and a symmetric matrix $P_\alpha > 0$ such that:
$$\left[\begin{array}{c|c} \begin{matrix} P_\alpha(A_\alpha - L_\alpha C_\alpha) \\ +C^T_\alpha C_\alpha \\ +(A_\alpha - L_\alpha C_\alpha)^T P_\alpha \end{matrix} & \begin{matrix} P_\alpha(E_{d,\alpha} \\ -L_\alpha F_{d,\alpha}) \\ +C^T_\alpha F_{d,\alpha} \end{matrix} \\ \hline \star & F^T_{d,\alpha} F_{d,\alpha} - \gamma^2_\alpha I \end{array}\right] < 0 \quad (19)$$
4) There exists a matrix U_α and a symmetric matrix $P_\alpha > 0$ such that:
$$\left[\begin{array}{c|c} \begin{matrix} P_\alpha A_\alpha + U_\alpha C_\alpha + C^T_\alpha C_\alpha \\ +A^T_\alpha P_\alpha + C^T_\alpha U_\alpha \end{matrix} & \begin{matrix} P_\alpha E_{d,\alpha} + U F_{d,\alpha} \\ +C^T_\alpha F_{d,\alpha} \end{matrix} \\ \hline \star & F^T_{d,\alpha} F_{d,\alpha} - \gamma^2_\alpha I \end{array}\right] < 0 \quad (20)$$
where the gain filter is $L_\alpha = -P^{-1}_\alpha U_\alpha$.
5) There exists a matrix U_α and a symmetric matrix $P_\alpha > 0$ such that:
$$\left[\begin{array}{c|c|c} \begin{matrix} P_\alpha A_\alpha + A^T_\alpha P_\alpha \\ +U_\alpha C_\alpha + C^T_\alpha U_\alpha \end{matrix} & \begin{matrix} P_\alpha E_{d,\alpha} \\ +U_\alpha F_{d,\alpha} \end{matrix} & C^T_\alpha \\ \hline \star & -\gamma^2_\alpha I & F^T_d \\ \hline \star & \star & -I \end{array}\right] < 0 \quad (21)$$
where the gain filter is $L_\alpha = -P^{-1}_\alpha U_\alpha$.

Proof 1:

1) The equivalence between (17) and (18) is straight forward from the definition of the $\mathcal{H}_\infty$ norm.
$$\|r_{d_\alpha}\|_2 < \gamma_\alpha \|d\|_2 \Leftrightarrow \frac{\|r_{d_\alpha}\|_2}{\|d\|_2} < \gamma_\alpha,$$
$$\|T_{rd_\alpha}(s)\|_\infty = \sup_{d \in \mathbb{R}^{n_d}} \frac{\|r_{d_\alpha}\|_2}{\|d\|_2} < \gamma_\alpha$$
2) The sufficient stability condition considering the candidate Lyapunov function: $V_\alpha|_{f=0} = \tilde{x}^T P_\alpha \tilde{x}$ is $\dot{V}_\alpha < 0$ and $P_\alpha > 0$.
Add to that, (17) is equivalent to $r^T_{d_\alpha} r_{d_\alpha} - \gamma^2_\alpha d^T d < 0$, then:
$$\dot{V}_\alpha + r^T_{d_\alpha} r_{d_\alpha} - \gamma^2_\alpha d^T d < 0$$
$$\Leftrightarrow \dot{\tilde{x}}^T P_\alpha \tilde{x} + \tilde{x}^T P_\alpha \dot{\tilde{x}} + r^T_{d_\alpha} r_{d_\alpha} - \gamma^2_\alpha d^T d < 0$$
In quadratic form:
$$\begin{bmatrix} \tilde{x} \\ d \end{bmatrix}^T \left[\begin{array}{c|c} \begin{matrix} P_\alpha A^*_\alpha + C^T_\alpha C_\alpha \\ +A^{*T}_\alpha P_\alpha \end{matrix} & \begin{matrix} P_\alpha B^*_{d_\alpha} \\ +C^T_\alpha F_{d_\alpha} \end{matrix} \\ \hline \star & F^T_{d_\alpha} F_{d_\alpha} - \gamma^2_\alpha I \end{array}\right] \begin{bmatrix} \tilde{x} \\ d \end{bmatrix} < 0$$

Then $\forall \begin{bmatrix} \tilde{x} \\ d \end{bmatrix} \neq 0$ it yields:

$$\left[\begin{array}{c|c} P_\alpha A^*_\alpha + C^T_\alpha C_\alpha \\ +A^{*T}_\alpha P_\alpha & P_\alpha B^*_{d_\alpha} \\ & +C^T_\alpha F_{d_\alpha} \\ \hline \star & F^T_{d_\alpha} F_{d_\alpha} - \gamma^2_\alpha I \end{array}\right] < 0$$

which is the same matrix as in (19).

3) The matrix inequality (19) is non linear. It can be transformed to LMI by using the definition $U_\alpha = -L_\alpha P_\alpha$.
4) The LMIs (20) and (21) are equivalent using Schur complement lemma.

Theorem 2: If there exist a sensible fault detection observer defined in (9)-(10), where F_{f_α} is full row rank, then the following expressions are equivalent:

1) There exist $\beta_\alpha \in \mathbb{R}, \beta_\alpha > 0$ such that the inequality holds:

$$\|r_{f_\alpha}\|_2 > \beta_\alpha \|f\|_2 \tag{22}$$

2) There exist $\beta_\alpha \in \mathbb{R}, \beta_\alpha > 0$ such that the inequality holds:

$$\|T_{rf_\alpha}(s)\|_- > \beta_\alpha \tag{23}$$

3) There exists a matrix L_α and a symmetric matrix P_α (not necessarily sign definite) such that:

$$\left[\begin{array}{c|c} P_\alpha(A_\alpha - L_\alpha C_\alpha) \\ +(A_\alpha - L_\alpha C_\alpha)^T P_\alpha & P_\alpha(E_{f_\alpha} - L_\alpha F_{f_\alpha}) \\ -C^T_\alpha C_\alpha & -C^T_\alpha F_{f_\alpha} \\ \hline \star & \beta^2_\alpha I - F^T_{f_\alpha} F_{f_\alpha} \end{array}\right] < 0 \tag{24}$$

4) There exists a matrix U and a matrix P_α such that:

$$\left[\begin{array}{c|c} -P_\alpha A_\alpha - U_\alpha C_\alpha + C^T_\alpha C_\alpha & -P_\alpha E_{f_\alpha} - U_\alpha F_{f_\alpha} \\ -A^T_\alpha P_\alpha - C^T_\alpha U_\alpha & +C^T_\alpha F_{f_\alpha} \\ \hline \star & F^T_{f_\alpha} F_{f_\alpha} - \beta^2_\alpha I \end{array}\right] > 0 \tag{25}$$

where the gain filter is $L_\alpha = -P^{-1}_\alpha U_\alpha$.

Proof 2: The equivalence between (23)-(26) could be easily demonstrated by following the same steps for the proofs of *Theorem 1*.

Remark 1: As it has been demonstrated in [9], P_α in the LMI for the $\mathcal{H}_-$ observer is not required to be sign definite, and this condition does not ensure the stability of the observer. However, joint $\mathcal{H}_-/\mathcal{H}_\infty$ observer is stable since it is guaranteed by *Theorem 1*: P_α is the same matrix in the LMI formulation, its sign definitiveness is thus imposed.

Theorem 3: For a given square $n \times n$ matrix A_α, if there exits a symmetric matrix $P_\alpha > 0$ and a diagonal $n \times n$ matrix $\Omega_{max,\alpha}$ such that the following inequality holds

$$A^T_\alpha P_\alpha + P_\alpha A_\alpha - 2\Omega_{max,\alpha} P_\alpha < 0 \tag{26}$$

Then all eigenvalues of A_α are on left plane of the eigenvalues of $\Omega_{max,\alpha}$

Proof 3: (26) is a result of a classical Lyapunov function for sufficient condition of stability. $\dot{x} = (A_\alpha - \Omega_{max,\alpha})x$ is stable if there exist a symmetric matrix $P > 0$ where $V = x^T P_\alpha x$, $\dot{V} < 0$. Thus

$$(A_\alpha - \Omega_{max,\alpha})^T P_\alpha + P_\alpha(A_\alpha - \Omega_{max,\alpha}) < 0 \tag{27}$$

which is equivalent to (26).

Corollary 1: If the pair (A_α, C_α) is observable, then there exits a symmetric matrix $P_\alpha > 0$, a matrix L_α and a diagonal matrix $\Omega_{max,\alpha}$ such that the following inequalities hold:

$$(A_\alpha - L_\alpha C_\alpha)^T P_\alpha + P_\alpha(A_\alpha - L_\alpha C_\alpha) - 2\Omega_{max,\alpha} P_\alpha > 0 \tag{28}$$

Then all eigenvalues of $(A_\alpha - L_\alpha C_\alpha)$ are assigned on left plane of the eigenvalues of $\Omega_{max,\alpha}$.

Theorem 4: Consider the disturbance to residual transfer function defined in (12), and assuming that there is no direct transfer of disturbances ($F_{d,\alpha} = 0$), then sufficient conditions to satisfy disturbance decoupling $T_{rd_\alpha}(s) = 0$ are:

(1) $Q_{r,\alpha} C_\alpha E_{d,\alpha} = 0$

(2) All rows of $H_\alpha = Q_{r,\alpha} C_\alpha$ are left eigenvectors of $A^*_\alpha = A_\alpha - L_\alpha C_\alpha$ corresponding to any eigenvalues.

Proof 4: (The α index is omitted in this proof for typing simplicity)

Let L and R left and right eigenvectors of the matrix A^*:

$$L = [L_1 \; ... \; L_n], \; R = [R_1 \; ... \; R_n]$$

By definition of left and right eigenvectors:

$$L^T R = \begin{bmatrix} L^T_1 R_1 & & 0 \\ & L^T_i R_i & \\ 0 & & L^T_n R_n \end{bmatrix}$$

$$L^T R = I; \; L^T = R^{-1}$$

Let D_Λ be the diagonal matrix of eigenvalues of A^*, it follows:

$$A^* = R D_\Lambda R^{-1}$$

$$e^{A^* t} = R e^{D_\Lambda t} R^{-1} = R e^{D_\Lambda t} L^T = \sum e^{\lambda_i t} R_i L^T_i$$

$$(sI - A^*)^{-1} = \mathcal{L}(e^{A^* t}) = \mathcal{L}\Big(\sum_{i=1}^{i=n} e^{\lambda_i t} R_i L^T_i\Big) = \sum_{i=1}^{i=n} \frac{R_i L^T_i}{s - \lambda_i}$$

$$T_{rd}(s) = \sum_{i=1}^{i=n} \frac{H R_i L^T_i E_d}{s - \lambda_i}$$

In order to satisfy $T_{rd}(s) = 0$, the denominator can be omitted:

$$H \sum_{i=1}^{i=n} R_i L^T_i E_d = H R L^T E_d = H E_d = Q_r C E_d = 0$$

$\square$

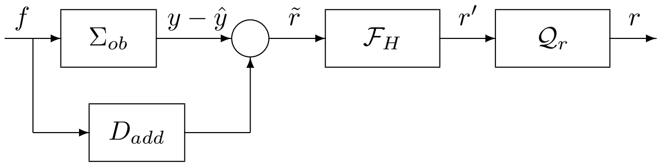

Fig. 1. $\mathcal{H}_-$ Loop Shaping with Additive filter weighting

IV. Fault detection observer design

A. $\mathcal{H}_-$ synthesis problem

It is easy to show that for strictly proper systems, where $D = 0$ or not of full row rank, the $\mathcal{H}_-$ index is always zero. In fact,

$$\lim_{\omega\to\infty} G(j\omega) = \lim_{\omega\to\infty}(C(j\omega I - A)^{-1}B + D) = D$$

Thus in strictly proper system, the $\mathcal{H}_-$ strategy cannot be used since L and Q_r have no effect in high frequencies. This is the case of actuators faults for example. Moreover, even for just proper systems where D is full column rank, the smallest gain over all frequency range β_α will always be restricted to the relation:

$$D^T D \geq \beta_\alpha^2 I$$

regardless the choice of L and Q_r.

To avoid this restriction two strategies could be adopted, first is multiplicative frequency weighting, and second is additive frequency weighting. These solutions have been presented in [17] and [9]. In this work the second solution is used.

The main purpose of additive filter is to translate the minimum gain to the low frequencies: it is the region where most faults are spectrally located (offsets...). This can be established by adding an auxiliary direct channel to the system, and then multiply by a high pass filter $\mathcal{F}_H$ as it is shown in figure (1) .

For $D_{f,\alpha} = \begin{bmatrix} D_{f1,\alpha} \\ 0 \end{bmatrix}$, a suitable matrix is $D_{add,\alpha} = \begin{bmatrix} 0 \\ \epsilon_\alpha I \end{bmatrix}$.

The high pass filter $\mathcal{F}_H$ is a weighting filter that is used to raise up the high-frequency response, so that minimum singular value of the whole system occurs near the low-frequency region. $\mathcal{F}_H$ has the following transfer function:

$$F_{H,\alpha}(s) = \left(\frac{s/\omega_{1,\alpha} + 1}{s/\omega_{2,\alpha} + 1}\right)^{m_\alpha} \tag{29}$$

where $\omega_{1,\alpha} < \omega_{2,\alpha}$ and m_α the order of the filter.

The parameters $\omega_{1,\alpha}$, $\omega_{2,\alpha}$ and m_α are chosen such that the transfer function $T_{rf_\alpha}(s)$ has the desired shape. This procedure is analog to the loop shaping method in the standard $\mathcal{H}_\infty$ problem.

The fault to residual transfer function to be used is:

$$\dot{\hat{x}} = A_\alpha \hat{x} + B_\alpha u + L_\alpha (y - \hat{y}) \tag{30}$$
$$\hat{y} = C_\alpha \hat{x} + D_\alpha u \tag{31}$$
$$\tilde{r}_\alpha = (y - \hat{y}) + D_{add,\alpha} f \tag{32}$$
$$\dot{x}_h = A_{h,\alpha} x_h + B_{h,\alpha} \tilde{r}_\alpha \tag{33}$$
$$r'_\alpha = C_{h,\alpha} x_h + D_{h,\alpha} \tilde{r}_\alpha \tag{34}$$
$$r_\alpha = Q_\alpha r'_\alpha \tag{35}$$

where $F_{H,\alpha}(s) := \left[\begin{array}{c|c} A_{h,\alpha} & B_{h,\alpha} \\ \hline C_{h,\alpha} & D_{h,\alpha} \end{array}\right]$ is a realization of the high pass filter $\mathcal{F}_{H_\alpha}$.

From equations (30)-(34), an augmented residual is deduced:

$$\begin{aligned} \begin{bmatrix} \dot{\tilde{x}} \\ \dot{x}_h \end{bmatrix} &= \begin{bmatrix} A_\alpha - L_\alpha C_\alpha & 0 \\ B_{h,\alpha} C_\alpha & A_{h,\alpha} \end{bmatrix} \begin{bmatrix} \tilde{x} \\ x_h \end{bmatrix} \\ &+ \begin{bmatrix} E_{d,\alpha} - L_\alpha F_{d,\alpha} \\ B_{h,\alpha} F_{d,\alpha} \end{bmatrix} d \\ &+ \begin{bmatrix} E_{f,\alpha} - L_\alpha F_{f,\alpha} \\ B_{h,\alpha}(F_{f,\alpha} + D_{add,\alpha}) \end{bmatrix} f \end{aligned} \tag{36}$$

$$\begin{aligned} r'_\alpha &= \begin{bmatrix} D_{h_\alpha} C_\alpha & C_{h_\alpha} \end{bmatrix} \begin{bmatrix} \tilde{x} \\ x_h \end{bmatrix} + \begin{bmatrix} D_{h,\alpha} F_{d,\alpha} \end{bmatrix} d \\ &+ \begin{bmatrix} D_{h,\alpha}(F_{f,\alpha} + D_{add,\alpha}) \end{bmatrix} f \end{aligned} \tag{37}$$

B. $\mathcal{H}_-/\mathcal{H}_\infty$ synthesis for augmented system

Theorems (1) and *(2)* are applied to the augmented systems:

$$G_{af,\alpha} := \left[\begin{array}{c|c} A_{a_\alpha} & B_{af_\alpha} \\ \hline C_{a_\alpha} & D_{af_\alpha} \end{array}\right], \quad G_{ad,\alpha} := \left[\begin{array}{c|c} A_{a_\alpha} & B_{ad_\alpha} \\ \hline C_{a_\alpha} & D_{ad_\alpha} \end{array}\right]$$

where

$$A_{a_\alpha} = \begin{bmatrix} A_\alpha - L_\alpha C_\alpha & 0 \\ B_{h,\alpha} C_\alpha & A_{h,\alpha} \end{bmatrix}, C_{a_\alpha} = \begin{bmatrix} D_{h_\alpha} C_\alpha & C_{h_\alpha} \end{bmatrix}$$

$$B_{af_\alpha} = \begin{bmatrix} E_{f,\alpha} - L_\alpha F_{f,\alpha} \\ B_{h,\alpha}(F_{f,\alpha} + D_{add,\alpha}) \end{bmatrix}, B_{ad_\alpha} = \begin{bmatrix} E_{d,\alpha} - L_\alpha F_{d,\alpha} \\ B_{h,\alpha} F_{d,\alpha} \end{bmatrix}$$

$D_{af_\alpha} = D_{h,\alpha}(F_{f,\alpha} + D_{add,\alpha})$ and $D_{ad_\alpha} = D_{h,\alpha} F_{d,\alpha}$.

Theorem 5: Consider $\mathcal{H}_-/\mathcal{H}_\infty$ fault detection observer for the augmented system in (37)-(38), for given positive reel scalars γ_α and β_α, there exist a matrix U_{a_α} and a symmetric matrix $P_{a_\alpha} > 0$ such that the following inequalities hold:

$$\left[\begin{array}{c|c} \begin{array}{c} P_{a_\alpha} A_{0_\alpha} + U_{a_\alpha} C_{0_\alpha} \\ +A_{0_\alpha}^T P_{a_\alpha} + C_{0_\alpha}^T U_{a_\alpha}^T \\ +C_{a_\alpha}^T C_{a_\alpha} \end{array} & \begin{array}{c} P_{a_\alpha} B_{0_\alpha} + U_{a_\alpha} F_{f,\alpha} \\ +C_{a_\alpha}^T D_{ad_\alpha} \end{array} \\ \hline \star & D_{ad_\alpha}^T D_{ad_\alpha} - \gamma_\alpha^2 I \end{array}\right] < 0 \tag{38}$$

$$\left[\begin{array}{c|c} \begin{array}{c} -P_{a_\alpha} A_{0_\alpha} - U_{a_\alpha} C_{0_\alpha} \\ -A_{0_\alpha}^T P_{a_\alpha} - C_{0_\alpha}^T U_{a_\alpha}^T \\ +C_{a_\alpha}^T C_{a_\alpha} \end{array} & \begin{array}{c} -P_{a_\alpha} B_{0_\alpha} \\ -U_{a_\alpha} F_{f,\alpha} \\ +C_{a_\alpha}^T D_{af_\alpha} \end{array} \\ \hline \star & D_{af_\alpha}^T D_{af_\alpha} - \beta_\alpha^2 I \end{array}\right] > 0 \tag{39}$$

where the gain filter is $L_\alpha = I_0^T (P_{a_\alpha})^{-1} U_{a_\alpha}$

and the matrices $A_{0_\alpha}, B_{f0_\alpha}, B_{d0_\alpha}, C_{0_\alpha}$, and I_0 are:

$$A_{0_\alpha} = \begin{bmatrix} A_\alpha & 0 \\ B_{h,\alpha} C_\alpha & A_{h,\alpha} \end{bmatrix}, C_{0_\alpha} =, I_0 = \begin{bmatrix} -I \\ 0 \end{bmatrix}$$

$$B_{f0_\alpha} = \begin{bmatrix} E_{f,\alpha} \\ B_{h,\alpha}(F_{f,\alpha} + D_{add,\alpha}) \end{bmatrix}, B_{d0_\alpha} = \begin{bmatrix} E_{d,\alpha} \\ B_{h,\alpha} F_{d,\alpha} \end{bmatrix}$$

Proof 5: Only the calculation to get inequality (38) are given here, the same steps are used to find (39).

Apply *Theorem 1* to the augmented system $G_{ad,\alpha}$. The deduced inequality is:

$$\left[\begin{array}{c|c} P_{a_\alpha}A_{a_\alpha}+C_{a_\alpha}^TC_{a_\alpha} & P_{a_\alpha}B_{ad_\alpha} \\ +A_{a_\alpha}^TP_{a_\alpha} & +C_{a_\alpha}^TD_{ad_\alpha} \\ \hline \star & D_{ad_\alpha}^TD_{ad_\alpha}-\gamma_\alpha^2I \end{array}\right]<0 \quad (40)$$

Then decomposing A_{a_α} and B_{ad_α} it follows:

$$P_{a_\alpha}A_{a_\alpha}=P_{a_\alpha}\begin{bmatrix} A_\alpha & 0 \\ B_{h,\alpha}C_\alpha & A_{h,\alpha}\end{bmatrix}+P_{a_\alpha}\begin{bmatrix}-I\\0\end{bmatrix}L_\alpha\begin{bmatrix}C_\alpha & 0\end{bmatrix}$$

$$P_{a_\alpha}B_{ad_\alpha}=P_{a_\alpha}\begin{bmatrix} E_{d,\alpha} \\ B_{h,\alpha}F_{d,\alpha}\end{bmatrix}+P_{a_\alpha}\begin{bmatrix}-I\\0\end{bmatrix}L_\alpha F_{d,\alpha}$$

Therefore, (40) becomes:

$$\left[\begin{array}{c|c} P_{a_\alpha}(A_{0_\alpha}+I_0L_\alpha C_{0_\alpha}) & P_{a_\alpha}(B_{0_\alpha} \\ +(A_{0_\alpha}+I_0L_\alpha C_{0_\alpha})^TP_{a_\alpha} & +I_0L_\alpha F_{d,\alpha}) \\ +C_{a_\alpha}^TC_{a_\alpha} & +C_{a_\alpha}^TD_{ad_\alpha} \\ \hline \star & D_{ad_\alpha}^TD_{ad_\alpha}-\gamma_\alpha^2I \end{array}\right]<0 \quad (41)$$

And by replacing $P_{a_\alpha}I_0L_\alpha$ by U_{a_α}, the BMI becomes the following LMI:

$$\left[\begin{array}{c|c} P_{a_\alpha}A_{0_\alpha}+U_\alpha C_{0_\alpha} & \\ +A_{0_\alpha}^TP_{a_\alpha}+C_{0_\alpha}^TU_\alpha & P_{a_\alpha}B_{0_\alpha}+U_\alpha F_{d,\alpha} \\ +C_{a_\alpha}^TC_{a_\alpha} & +C_{a_\alpha}^TD_{ad_\alpha} \\ \hline \star & D_{ad_\alpha}^TD_{ad_\alpha}-\gamma_\alpha^2I \end{array}\right]<0 \quad (42)$$

□

C. Disturbance decoupling

Even though disturbance attenuation could be achieved by minimizing the $\mathcal{H}_\infty$ norm, it could be enhanced if the sufficient conditions of *Theorem (4)* are satisfied.

An easy solution with one degree of freedom for the first condition $Q_{r,\alpha}C_\alpha E_{d,\alpha}=0$ is to calculate $Q_{r,\alpha}$ using the Moore-Penrose pseudoinverse of the full column-rank matrix $C_\alpha E_{d,\alpha}$:

$$(C_\alpha E_{d,\alpha})^+ = \left[(C_\alpha E_{d,\alpha})^T(C_\alpha E_{d,\alpha})\right]^{-1}(C_\alpha E_{d,\alpha})^T \quad (43)$$

$$Q_{r,\alpha} = Q_{x,\alpha}\left[I-C_\alpha E_{d,\alpha}(C_\alpha E_{d,\alpha})^+\right] \quad (44)$$

D. Dynamic fault detection observer (DFDO) design

The algorithm to design the DRFDO is the following:

(1) Choose suitable $D_{add,\alpha}$ matrices, and the high pass filters $\mathcal{F}_{H_\alpha}$.

(2) Choose the desired dynamics of the observer, it is set by the pole region from $\Omega_{max,\alpha}$.

(3) Solve the following LMIs, minimizing $\beta_\alpha^2-\gamma_\alpha^2$ to get the optimal value of the gain matrices L_α.

$$\left[\begin{array}{c|c} P_{a_\alpha}A_{0_\alpha}+U_{a_\alpha}C_{0_\alpha} & \\ +A_{0_\alpha}^TP_{a_\alpha}+C_{0_\alpha}^TU_{a_\alpha}^T & P_{a_\alpha}B_{0_\alpha}+U_{a_\alpha}F_{f,\alpha} \\ +C_{a_\alpha}^TC_{a_\alpha} & +C_{a_\alpha}^TD_{ad_\alpha} \\ \hline \star & D_{ad_\alpha}^TD_{ad_\alpha}-\gamma_\alpha^2I \end{array}\right]<0$$

$$\left[\begin{array}{c|c} -P_{a_\alpha}A_{0_\alpha}-U_{a_\alpha}C_{0_\alpha} & -P_{a_\alpha}B_{0_\alpha} \\ -A_{0_\alpha}^TP_{a_\alpha}-C_{0_\alpha}^TU_{a_\alpha}^T & -U_{a_\alpha}F_{f,\alpha} \\ +C_{a_\alpha}^TC_{a_\alpha} & +C_{a_\alpha}^TD_{af_\alpha} \\ \hline \star & D_{af_\alpha}^TD_{af_\alpha}-\beta_\alpha^2I \end{array}\right]>0$$

$$P_{a_\alpha}A_{0_\alpha}+A_{0_\alpha}^TP_{a_\alpha}+U_{a_\alpha}C_{0_\alpha}+C_{0_\alpha}^TU_{a_\alpha}^T<0$$

$$P_{a_\alpha}>0$$

The gain matrix $L_\alpha=I_0^T(P_{a_\alpha})^{-1}U_{a_\alpha}$

(4) Calculate the post-matrices gain $Q_{r,\alpha}$ for disturbance decoupling as in (43)-(44):

$$Q_{r,\alpha}=Q_{x,\alpha}\left[I-C_\alpha E_{d,\alpha}(C_\alpha E_{d,\alpha})^+\right] \quad (45)$$

V. Example

Consider the LTI switched MIMO system of the form (1)-(2) with the following matrices:

$$A_1=\begin{bmatrix}-20 & -100\\ 1 & 0\end{bmatrix}, E_{d_1}=\begin{bmatrix}0.1\\0.2\end{bmatrix},\ E_{f_1}=\begin{bmatrix}0.1\\0.01\end{bmatrix},$$
$$C_1=\begin{bmatrix}-40 & -300\\ -15 & -92\end{bmatrix},\ F_{d_1}=\begin{bmatrix}0.1\\0\end{bmatrix},\ F_{f_1}=\begin{bmatrix}0\\1\end{bmatrix}.$$

$$A_2=\begin{bmatrix}-23 & -110\\ 1.4 & 3\end{bmatrix}, E_{d_2}=\begin{bmatrix}0.225\\0.54\end{bmatrix},\ E_{f_2}=\begin{bmatrix}0.27\\0.051\end{bmatrix},$$
$$C_2=\begin{bmatrix}-40 & -290\\ -14 & -97\end{bmatrix},\ F_{d_2}=\begin{bmatrix}0.3\\0\end{bmatrix},\ F_{f_2}=\begin{bmatrix}0\\1.4\end{bmatrix}.$$

The system is observable, and just proper. A stable $\mathcal{H}_-/\mathcal{H}_\infty$ FD observer can be designed.

First, define the loop shaping matrices and weighting filter: suitable $D_{add,\alpha}$ matrices are chosen for the additive filter. Let

$$D_{add,1}=\begin{bmatrix}.06 & .001\end{bmatrix}^T, D_{add,2}=\begin{bmatrix}.005 & .0002\end{bmatrix}^T$$

$$\text{and}\quad F_{H,1}(s)=F_{H,2}(s)=\left(\frac{s/.095+1}{s/8+1}\right)^2$$

Then, the matrices for minimum poles assignment are:

$$\Omega_{max,1}=\Omega_{max,2}=-3I$$

Using Matlab optimization tools such YALMIP or SeDuMi, the set of LMIs is then solved minimizing the criterion $\beta_\alpha^2-\gamma_\alpha^2$. The post gain matrices Q_{r_α} are calculated using equation (45).

In result, the values of L_α and Q_{r_α} are:

$$L_1=\begin{bmatrix}-0.3269 & 1.9446\\ 0.0643 & -0.2372\end{bmatrix}, L_2=\begin{bmatrix}-0.3269 & 1.9446\\ 0.0643 & -0.2372\end{bmatrix}$$

$$Q_{r_1}=\begin{bmatrix}0.0882 & -0.2835\\ -0.2835 & 0.9118\end{bmatrix}, Q_{r_2}=\begin{bmatrix}0.0924 & -0.2896\\ -0.2896 & 0.9076\end{bmatrix}$$

The poles of the observer are

$$\lambda_{1,1}=-3.6394,\ \lambda_{2,1}=-22.7884,$$
$$\lambda_{2,1}=-3.0913,\ \lambda_{2,1}=-123.796$$

The dashed blue curves in figure (2) are for the open loop residual to fault/disturbance sensitivity. The observer

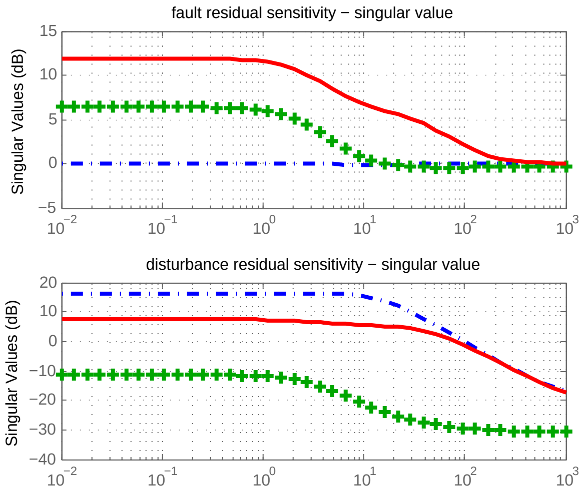

Fig. 2. Gain plot of $T_{rf_1}(j\omega)$(top) and $T_{rd_1}(j\omega)$ (bottom)

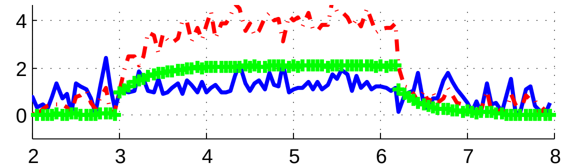

Fig. 3. Residuals obtained using different stages of observer design

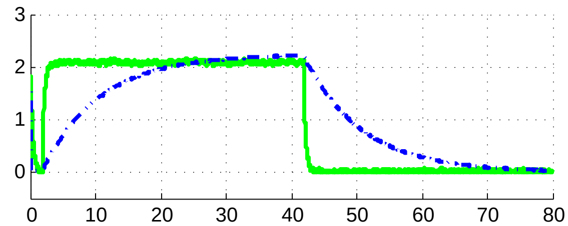

Fig. 4. Residuals whith and without eigen values assignment constraint

got from the solution of LMIs without decoupling gives the results shown in the red solid line. The third curve green asterisks is obtained with the full design: disturbance decoupling is achieved when post multiplying by Qr_1.

These gain plots shows that with this design, the sensitivity of residuals toward faults has been enhanced, and the disturbance effect have been simultaneously reduced.

Considering the time analysis, figure (3) shows the residual obtained using the designed observer: the original residual signal is in solid blue. In dashed red, the residual got using $\mathcal{H}_-/\mathcal{H}_\infty$ with eigen value assignment. The third curve in asterisk is the plot of the full observer, designed with the decoupling matrix. On figure (4), the standard $\mathcal{H}_-/\mathcal{H}_\infty$ observer is plotted in dashed blue, it has a low response (20 s). The third constraint (cf. LMI (28)) of our multi-objective formulation insures the fast response of the observer.

VI. Conclusion and further work

In this paper, we have studied the $\mathcal{H}_-$ index problem for fault detection, and we have investigated the multiobjective design of $\mathcal{H}_-/\mathcal{H}_\infty$ with eigenvalue assignment observers for switched systems.

This three objectives design ensures a higher sensibility of the residuals towards the faults, a disturbance attenuation and decoupling, and a dynamics for varying fault detection. Sufficient conditions are given. We propose a compromise of these objectives as a criterion to minimize, and then we formulate it as an LMIs feasibility problem. By using efficient LMI solver, solution of the optimization problem can be found.

Furthermore, the $\mathcal{H}_-$ index design has been studied using a loop shaping approach. It can be applied for either actuator or sensor fault detection. Finally, we illustrated the effectiveness of the proposed design by a numerical application. The designed observer has reached the desired objectives .

For future work, the design of this observer with mutli-objective formulation can be extended for robust fault detection in uncertain switched system, using the same framework of the classical $\mathcal{H}_\infty$ design.

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